D1 (H) Maths

paper-I

Trigonometry

Demoivre's theorem

Statement:- If n is a positive or negative integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and if n is a fraction, positive or negative, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ (R)

Expansion of cos x

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots$$ (R) [R means remember]

Expansion of sin x

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$$ [Remember]

Expansion of tan x

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$ (R)

Imp.

Complex Argument

(i.) Prove $e^{ix} = \cos x + i\sin x$

$$\because e^z = 1 + \frac{z}{\lfloor 1} + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots$$

putting $z = ix$

$$e^{ix} = 1 + \frac{ix}{\lfloor 1} + \frac{i^2x^2}{\lfloor 2} + \frac{i^3x^3}{\lfloor 3} + \frac{i^4x^4}{\lfloor 4} + \cdots$$

$\therefore e^{ix} = 1 + \frac{ix}{\lfloor 1} - \frac{x^2}{\lfloor 2} - \frac{ix^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{ix^5}{\lfloor 5} - \frac{x^6}{\lfloor 6} + \cdots$

or $e^{ix} = \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots\right) + i\left(\frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right)$

or $e^{ix} = \cos x + i\sin x$ (1) (R)

Similarly

$e^{-ix} = \cos x - i\sin x$ (2) (R)

Adding $\therefore$ $e^{ix} + e^{-ix} = 2\cos x$

$\therefore \cos x = \frac{e^{ix} + e^{-ix}}{2}$ (3) (Remember)

Again $e^{ix} - e^{-ix} = 2i\sin x$

$\therefore \sin x = \frac{e^{ix} - e^{-ix}}{2i}$ (4) (R)

$e^{2n\pi i} = \cos 2n\pi + i\sin 2n\pi = 1 + 0 = 1$ (5) (R)

put $\log(\alpha + i\beta)$ in the form $A + iB$

gmp.

Let $\alpha = r\cos\theta$, $\beta = r\sin\theta$

$\therefore \alpha^2 + \beta^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore r = \sqrt{\alpha^2 + \beta^2}$

$\frac{\beta}{\alpha} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\therefore \theta = \tan^{-1} \beta/\alpha$

3 Now $\log(\alpha+i\beta) = \log\{r\cos\theta + ir\sin\theta\}$

$= \log r(\cos\theta + i\sin\theta)\times 1$

$= \log r.e^{i\theta}.e^{2n\pi i}$ $[\because e^{i2n\pi}=1]$

$\therefore \log(\alpha+i\beta) = \log r.e^{i\theta+2n\pi i}$

$= \log r + \log e^{(2n\pi+\theta)i}$

$= \log\sqrt{\alpha^2+\beta^2} + (2n\pi+\theta)i\log e$

$[\log m^n = n\log m]$

$\therefore \log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + (2n\pi+\tan^{-1}\frac{\beta}{\alpha})i$

principal value of $\log(\alpha+i\beta)$

we put $n=0$

$\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$ (R)

Imp. **Gregory's series**

If $-\pi/4 \le \theta \le \pi/4$

then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$

proof:- we know

$\log(1+ix) = \frac{1}{2}\log(1^2+x^2) + i\tan^{-1}\frac{x}{1}$ —(1)

we also know

$\log(1+ix) = ix - \frac{1}{2}i^2x^2 + \frac{1}{3}i^3x^3 - \frac{1}{4}i^4x^4 + \dots$

$= ix + \frac{1}{2}x^2 - \frac{1}{3}ix^3 - \frac{1}{4}x^4 + \dots$

or $\log(1+ix) = (\frac{1}{2}x^2 - \frac{1}{4}x^4 + \dots) + i(x - \frac{x^3}{3} + \dots)$ —(2)

(4) From (1) and (2) Equating imaginary part, we have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \quad \text{---} (3) \quad (R)$$

Let $\tan^{-1}x = \theta$ $\therefore x = \tan\theta$

from (3)

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots$$

Hyperbolic functions:

(1) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(3) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(4) $e^x + e^{-x} = 2\cosh x$

$e^x - e^{-x} = 2\sinh x$

$\therefore 2e^x = 2(\cosh x + \sinh x)$

or $e^x = \cosh x + \sinh x$

$e^{-x} = \cosh x - \sinh x$

(5) Prove $\cosh^2 x - \sinh^2 x = 1$

$$\text{LHS} = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \tfrac{1}{4}\left[e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x e^{-x})\right]$$

$$= \tfrac{1}{4}\left[\cancel{e^{2x}} - \cancel{e^{2x}} + \cancel{e^{-2x}} - \cancel{e^{-2x}} + 2 - 2\right]$$

$$= \tfrac{1}{4}[4] = 1 \quad \text{Proved}$$

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(6.) prove $\cosh x = \cos(ix)$

$$L.H.S = \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\text{or } \cosh x = \frac{e^{-i\,ix} + e^{i\cdot ix}}{2}$$

$$= \frac{e^{i(ix)} + e^{-i(ix)}}{2}$$

$\cosh x = \cos(ix)$ proved

(7) prove $\sinh x = -i\sin(ix)$

$$L.H.S = \sinh x = \frac{e^{x} - e^{-x}}{2} = \frac{e^{-i(ix)} - e^{i(ix)}}{2}$$

$$\sinh x = -i\left[\frac{e^{i(ix)} - e^{-i(ix)}}{2i}\right]$$

$$= -i\sin(ix)$$

$$\sin ix = -\frac{1}{i}\sinh x = i\sinh x \quad (R)$$

Similarly $\tanh x = \dfrac{\sinh x}{\cosh x}$

(6)

$$= \frac{-i\sin ix}{\cos ix}$$

$$= -i(\tan ix) \quad (R)$$

~~Sin hx~~ $\sin hx = -i\sin ix \therefore i\sin hx = \sin ix$

$$\boxed{\begin{aligned} \sin hx &= -i\sin(ix) \\ \sin ix &= i\sinh x \end{aligned}}$$

(R)

(8) $\cos hx = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots$

L.H.S $\cosh x = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left[\left\{1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\right\} + \left\{1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} - \cdots\right\}\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{\lfloor 2} + \frac{2x^4}{\lfloor 4} + \cdots\right]$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots\right]$$

(9)

$\sin hx = \frac{e^x - e^{-x}}{2}$

$$= \frac{1}{2}\left[\left\{1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\right\} - \left\{1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \cdots\right\}\right]$$

$$= \frac{1}{2}\left[2\frac{x}{\lfloor 1} + \frac{2x^3}{\lfloor 3} + \cdots\right]$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[\frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots\right]$$